# Functional Latent Block Model

for functional data co-clustering

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#### The data

- electricity consumption measured by Linky meters for EDF
- 27 millions of customers / 730 daily consumption over 2 years

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Figure: Sample of 20 consumptions for 20 days

### The data

- ► large data matrix  $\mathbf{x} = (x_{ij}(t))_{1 \le i \le n, 1 \le j \le p}$
- there is a need to summarize this data flow
- both n and p are (very) large
- ⇒ need for clustering of row (customers) and column (days of consumption):

need for co-clustering of functional data

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## Co-clustering ?

#### Simultaneous clustering of rows (individuals) and column (features)



coclustering result



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legend: color level =  $\frac{1}{T} \int_T x_{ij}(t)$ 

### Electricity consumption = functional data

- ► x<sub>ij</sub>(t) are not totally known but only observed at a finite number of times points x<sub>ij</sub>(t<sub>1</sub>), x<sub>ij</sub>(t<sub>2</sub>),...
- need to reconstruct the functional nature of data
- $\Rightarrow$  basis expansion assumption:

$$x_{ij}(t) = \sum_{h=1}^m a_{ijh}\phi_h(t), \quad t \in [0, T].$$

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where  $(\phi_h(t))_h$ : spline, Fourier, wavelets...

*a<sub>ijh</sub>* estimated by least square smoothing



The fLBM model

Inference with SEM-Gibbs algorithm

Numerical experiments

Application on EDF consumption curves

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## Latent Block Model (LBM)

#### Assumptions

- row z = (z<sub>ik</sub>)<sub>i,k</sub> and column w = (w<sub>hℓ</sub>)<sub>h,ℓ</sub> partitions are independent
- conditionally on (z, w), x<sub>ij</sub> are independent and generated by a block-specific distribution:



### Latent Block Model

#### Latent Block Model (LBM)

 $n \times d$  random variables **x** are assumed to be independent once the row **z** =  $(z_{ik})_{i,k}$  and column **w** =  $(w_{h\ell})_{h,\ell}$  partitions are fixed:

$$p(\mathbf{x}; \theta) = \sum_{\mathbf{z} \in V} \sum_{\mathbf{w} \in W} p(\mathbf{z}; \theta) p(\mathbf{w}; \theta) p(\mathbf{x} | \mathbf{z}, \mathbf{w}; \theta)$$

with

V (W) set of possible partitions of rows (column) into K (L) groups,

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• 
$$p(\mathbf{z}; \theta) = \prod_{ik} \alpha_k^{z_{ik}} \text{ and } p(\mathbf{w}; \theta) = \prod_{h\ell} \beta_{\ell}^{w_{h\ell}}$$

• 
$$p(\mathbf{x}|\mathbf{z}, \mathbf{w}; \theta) = \prod_{ijk\ell} p(\mathbf{a}_{ij}; \theta_{k\ell})^{v_{ik}w_{h\ell}}$$

$$\blacktriangleright \ \theta = (\alpha_k, \beta_\ell, \theta_{k\ell})$$

### The functional Latent Block Model (fLBM)

 $p(\mathbf{a}_{ij}; \theta_{k\ell})$  is the funHDDC distribution (Bouveyron & Jacques, ADAC, 2011):

$$\mathbf{a}_{ij}|(\mathbf{z}_{ik}=1,\mathbf{w}_{j\ell}=1)\sim\mathcal{N}(U_{k\ell}\mu_{k\ell},U_{k\ell}\Sigma_{k\ell}U_{k\ell}^{t}+\Xi_{k\ell})$$

where

- ► U<sub>kℓ</sub> projects the **a**<sub>ij</sub> into a low dimensional subspace for block kℓ
- $(\mu_{k\ell}, \Sigma_{k\ell})$ : (mean, variance) into the low-dimensional subspace,

with  $s_{k\ell j} > b_{k\ell}$  for all j = 1, ..., d.



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### LBM inference

#### LBM inference

The aim is to estimate θ by maximizing the observed log-likelihood

$$\ell(\theta; \mathbf{x}) = \sum_{\mathbf{v}, \mathbf{w}} \ln p(\mathbf{x}, \mathbf{v}, \mathbf{w}; \theta).$$

where functional data  $\mathbf{x}$  are represented by their coefficient  $\mathbf{a}$ , and  $\mathbf{v}$  and  $\mathbf{w}$  are missing row and column partitions

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- EM is not computationally tractable
- $\blacktriangleright$   $\Rightarrow$  variational or stochastic version should be used

### SEM-Gibbs algorithm for LBM inference

- ▶ init : θ<sup>(0)</sup>, w<sup>(0)</sup>
- SE step
  - generate the row and column parititon (v<sup>(q+1)</sup>, w<sup>(q+1)</sup>) using a Gibbs sampling
- M step
  - Estimate  $\theta$ , conditionally on  $\mathbf{v}^{(q+1)}, \mathbf{w}^{(q+1)}$  obtained at the SE step.

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### SEM-Gibbs: SE step

1. generate the row partition  $z_i^{(q+1)} = (z_{i1}^{(q+1)}, \dots, z_{iK}^{(q+1)}) | \mathbf{a}, \mathbf{w}^{(q)}$  for all  $1 \le i \le n$  according to  $z_i^{(q+1)} \sim \mathcal{M}(1, \tilde{z}_{i1}, \dots, \tilde{z}_{iK})$  with for  $1 \le k \le K$ 

$$\tilde{z}_{ik} = \mathsf{p}(z_{ik} = 1 | \mathbf{a}, \mathbf{w}^{(q)}; \theta^{(q)}) = \frac{\alpha_k^{(q)} f_k(\mathbf{a}_i | \mathbf{w}^{(q)}; \theta^{(q)})}{\sum_{k'} \alpha_{k'}^{(q)} f_{k'}(\mathbf{a}_i | \mathbf{w}^{(q)}; \theta^{(q)})}$$

where 
$$\mathbf{a}_i = (\mathbf{a}_{ij})_j$$
 and  $f_k(\mathbf{a}_i | \mathbf{w}^{(q)}; \theta^{(q)}) = \prod_{j\ell} p(\mathbf{a}_{ij}; \theta_{k\ell}^{(q)})^{\mathbf{w}_{j\ell}^{(q)}}$ ,

2. generate the column partition  $w_j^{(q+1)} = (w_{j1}^{(q+1)}, \dots, w_{jL}^{(q+1)}) | \mathbf{a}, \mathbf{z}^{(q+1)}$  for all  $1 \le j \le p$  according to  $w_j^{(q+1)} \sim \mathcal{M}(1, \tilde{w}_{j1}, \dots, \tilde{z}_{jL})$  with for  $1 \le \ell \le L$ 

$$\tilde{w}_{j\ell} = p(w_{j\ell} = 1 | \mathbf{a}, \mathbf{z}^{(q+1)}; \theta^{(q)}) = \frac{\beta_{\ell}^{(q)} f_{\ell}(\mathbf{a}_j | \mathbf{z}^{(q+1)}; \theta^{(q)})}{\sum_{\ell'} \beta_{\ell'}^{(q)} f_{\ell'}(\mathbf{a}_j | \mathbf{z}^{(q+1)}; \theta^{(q)})}$$

where  $f_{\ell}(\mathbf{x}_{j}|\mathbf{z}^{(q+1)};\theta^{(q)}) = \prod_{ik} p(\mathbf{a}_{ij};\theta^{(q)}_{k\ell})^{z_{ik}^{(q+1)}}$ .

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### SEM-Gibbs: M step

same M step than for FunHDDC (Bouveyron & Jacques, ADAC, 2011):

• 
$$\alpha_k^{(q+1)} = \frac{1}{n} \sum_i z_{ik}^{(q+1)}$$
 and  $\beta_\ell^{(q+1)} = \frac{1}{p} \sum_j w_{j\ell}^{(q+1)}$ ,

•  $\mu_{k\ell}^{(q+1)} = \frac{1}{n_{k\ell}^{(q+1)}} \sum_{i} \sum_{j} \mathbf{a}_{ij}^{z_{ik}^{(q+1)}} w_{j\ell}^{(q+1)}$  with  $n_{k\ell}^{(q+1)} = \sum_{i} \sum_{j} z_{ik}^{(q+1)} w_{j\ell}^{(q+1)}$ ,

• for the model parameters  $s_{k\ell j}$ ,  $b_{k\ell}$  and  $Q_{k\ell j}$ :

- *d* first columns of  $Q_k$ : first eigenvectors of  $\Omega^{\frac{1}{2}} C_{k\ell}^{(q)} \Omega^{\frac{1}{2}}$ ,
- $s_{k\ell j}, j = 1, ..., d$ : largest eigenvalues of  $\Omega^{\frac{1}{2}} C_{k\ell}^{(q)} \Omega^{\frac{1}{2}}$ ,
- $b_k$ : trace $(\Omega^{\frac{1}{2}} C_{k\ell}^{(q)} \Omega^{\frac{1}{2}}) \sum_{j=1}^d s_{k\ell j}^{(q)}$ ,

where  $C_{k\ell}^{(q)}$  is the sample covariance matrix of block  $k\ell$ :

$$C_{k\ell}^{(q)} = \frac{1}{n_{k\ell}^{(q)}} \sum_{i=1}^{n} \sum_{j=1}^{p} Z_{ik}^{(q+1)} \omega_{j\ell}^{(q+1)} (\mathbf{a}_{ij} - \mu_{k\ell}^{(q)})^{t} (\mathbf{a}_{ij} - \mu_{k\ell}^{(q)}),$$

and  $\Omega = (\omega_{jk})_{1 \le j,k \le m}$  with  $\omega_{jk} = \int_0^T \phi_j(t)\phi_k(t)dt$ .

### LBM inference

#### SEM-Gibbs algorithm for LBM inference

•  $\hat{\theta}$  is obtained by mean of the sample distribution (after a burn in period)

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Final bipartition  $(\hat{\mathbf{v}}, \hat{\mathbf{w}})$  estimated by MAP conditionally on  $\hat{\theta}$ 

### LBM inference

#### Choosing K and L

We use the ICL-BIC criterion developed in (Lomet 2012) for continuous data co-clustering. Thus, K and L can be chosen by maximizing

$$\mathsf{ICL}\mathsf{-}\mathsf{B}\mathsf{IC}(K,L) = \log \mathsf{p}(\mathbf{x}, \hat{\mathbf{v}}, \hat{\mathbf{w}}; \hat{\theta}) - \frac{K-1}{2} \log n - \frac{L-1}{2} \log p - \frac{KL\nu}{2} \log(np)$$

where  $\nu = md + d + 1$  is the number of continuous parameters per block and

$$\log \mathsf{p}(\mathbf{x}, \hat{\mathbf{v}}, \hat{\mathbf{w}}; \hat{\theta}) = \prod_{ik} \hat{z}_{ik} \log \alpha_k + \prod_{j\ell} \hat{w}_{j\ell} \log \beta_\ell + \sum_{ijk\ell} \hat{z}_{ik} \hat{w}_{j\ell} \log \mathsf{p}(\mathbf{a}_{ij}; \hat{\theta}_{k\ell}).$$

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### Simulation setting

•  $f_1(t), ..., f_4(t)$  are defined as block means



all curves are sampled as follows:

$$X_{ij}(t)|Z_{ik}W_{jl} = 1 \sim \mathcal{N}(\mu_{kl}(t), \sigma^2),$$

where  $\sigma = 0.3$ ,  $\mu_{11} = \mu_{21} = \mu_{33} = \mu_{42} = f_1$ ,  $\mu_{12} = \mu_{22} = \mu_{31} = f_2$ ,  $\mu_{13} = \mu_{32} = f_3$  and  $\mu_{23} = \mu_{41} = \mu_{43} = f_4$ .

noise is added by adding \(\tag{\varsigma}\) of curves from other blocks.

### 3 scenarios of simulation

Table: Parameter values for the three simulation scenarios.

Scenario	A	В	С								
n (nb. of rows)		100									
p (nb. of columns)	100										
T (length of curves)	30										
K (row groups nb.)	3	4	4								
L (col. groups nb.)	3	3	3								
$\alpha$ (row group prop.)	(0.333,, 0.333)	(0.2, 0.4, 0.1, 0.3)	(0.2, 0.4, 0.1, 0.3)								
$\beta$ (col. group prop.)	(0.333,, 0.333)	(0.4, 0.3, 0.3)	(0.4, 0.3, 0.3)								
au (simulation noise)	0	0.1	0.3								

### ICL performance for choosing (K, L)

Scenario A ( $K = 3, L = 3$ )							Scenario B ( $K = 4, L = 3$ )								
$K \setminus L$	1	2	3	4	5	6		K	$\setminus L$	1	2	3	4	5	6
1	0	0	0	0	0	0		1	1	0	0	0	0	0	0
2	0	0	0	0	0	0		2		0	0	0	0	0	0
3	0	0	100	0	0	0		3	3	0	0	0	0	0	0
4	0	0	0	0	0	0		4	1	0	0	70	0	1	0
5	0	0	0	0	0	0		Ę	5	0	0	26	1	0	0
6	0	0	0	0	0	0		6		0	0	2	0	0	0
Scenario C ( $K = 4, L = 3$ )										]					
			$K \setminus$	Q	1	2		3	4	5	6	1			
			1		0	0		0	0	0	0	]			
2 3 4			0	0		17	0	0	0	]					
		3		0	0		77	0	0	0	]				
		4		0	0		5	0	0	0	]				
			5		0	0		1	0	0	0	]			
			6		0	0		0	0	0	0	]			

### Co-clustering results for scenario B

True partition



FunLBM partition



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## Co-clustering results



Figure: Adjusted Rand index values for the different initialization procedures on the three simulation scenarios.

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### The data



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- electricity consumption measured by Linky meters for EDF
- 27 millions of customers / 730 daily consumption over 2 years

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Figure: Sample of 20 consumptions for 20 days

# ICL values (choice of (K, L))



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# Clustering of columns (dates)



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### Average consumption curves of each block



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### Geographical clusters distributions



Figure: Proportions on households per French departments in each of the 10 clusters found by FunLBM.

### Conclusions

### Results

- real data application needs development of a co-clustering algorithm for functional data
- co-clustering algorithm has been developed based on a functional Latent Block model
- numerical experiments show the efficiency of SEM-Gibbs for model estimation as well as ICL-BIC for selecting of the number of blocks
- Results on EDF data are significant

#### References

- Bouveyron, C. and Jacques, J. (2011), Model-based Clustering of Time Series in Group-specific Functional Subspaces, Advances in Data Analysis and Classification, 5[4], 281-300.
- ▶ Govaert, G. and Nadif, M. (2013). Co-Clustering. Wiley-ISTE.